Cantor on Frege's *Foundations of Arithmetic*

Cantor's 1885 review of Frege's *Die Grundlagen der Arithmetik*

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In 1885, Georg Cantor published his review of Gottlob Frege's *Grundlagen der Arithmetik*. In this essay we provide its first English translation together with an introductory note. We also provide a translation of a note by Ernst Zermelo on Cantor's review, and a new translation of Frege's brief response to Cantor.

In recent years it has entered philosophical folklore that Cantor's 1885 review of Frege's *Grundlagen* already contained a warning to Frege. This warning is said to concern the defectiveness of Frege's notion of extension. The exact scope of such speculations vary and sometimes extend as far as crediting Cantor with an early hunch of the paradoxical nature of Frege's notion of extension. William Tait goes even further and deems Frege 'reckless' for having missed Cantor's *explicit* warning regarding the notion of extension. As such, Cantor's purported inkling would have predated the discovery of the Russell-Zermelo paradox by almost two decades.

In our introductory essay we discuss this alleged implicit (or even explicit) warning, separating two issues: first, whether the most natural reading of Cantor's criticism provides an indication that the notion of extension is defective; second, whether there are other ways of understanding Cantor that support such an interpretation and can serve as a precisification of Cantor's presumed warning.

Introduction

In 1885, Georg Cantor published his review of Gottlob Frege's *Grundlagen der Arithmetik* in the journal *Deutsche Litteraturzeitung*.[1] The review was later reprinted in Cantor's *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, edited by Ernst Zermelo *Cantor 1932*. In this essay we provide its

[1]See *Cantor 1885* and *Frege 1884*.

first English translation. Further included are the first English translation of Zermelo's note *Zermelo 1932* on Cantor's review, and a new English translation of Frege's brief response to Cantor which was published two months later in the same journal,[2] probably as a paid ad.[3]

These three short pieces are of historical interest in the foundations of mathematics and early analytic philosophy. Cantor's review, despite being in praise of the critical parts of Frege's *Grundlagen*, finds fault with Frege's own proposal. In particular, he objects to Frege's attempt to define 'number' by means of 'extension'. Frege's logicist project, as first outlined in the *Grundlagen der Arithmetik*, attempted to reduce arithmetic to pure logic. Although in the *Grundlagen* Frege believes the appeal to extensions to be dispensable to his project,[4] it plays a crucial role in the later execution of the project in his *Grundgesetze der Arithmetik*, *1893/1903*. There Frege characterises extensions by means of the now infamous Basic Law V:

$$\vdash\!\!-\, (\grave{\varepsilon} f(\varepsilon) = \grave{\alpha} g(\alpha)) = (-\!\overset{\mathfrak{a}}{\smile}\!-\, f(\mathfrak{a}) = g(\mathfrak{a}))$$

which in Peano-esque notation might be rendered as:

$$\forall F \forall G(\varepsilon x \!:\! Fx = \varepsilon x \!:\! Gx \;\equiv\; \forall x(Fx \equiv Gx))$$

and which states that the extension of F equals the extension of G if and only if F and G are co-extensional. As intuitive as this may sound, in *1902* Bertrand Russell informed Frege in a personal letter of the paradox to which Basic Law V is subject. It was independently discovered by Zermelo around 1899–1900 in relation to the naïve conception of set:[5] consider the set of all and only those sets that are not members of themselves. This set is a member of itself if and only if it is not, as a moment's thought will show. This naïve conception of set is thus inconsistent. The paradox arises in exactly the analogous way for Frege's Basic Law V using extensions instead of sets.[6] In response, Frege offers a rescue attempt in the postscript to volume II of his *Grundgesetze*. Frege's way out, however, does not in fact avoid the paradox but merely prolongs its derivation, as it is now known. Frege realised this himself later on and subsequently abandoned his logicist project.

In recent years, however, it has entered philosophical folklore that Cantor's *1885* review of Frege's *Grundlagen* already contained a warning to Frege. This warning is said to concern the defectiveness of Frege's notion of extension. The exact scope of such speculations vary and sometimes extend as far as crediting Cantor with an early hunch of the paradoxical nature of Frege's notion of extension. William

[2] See *Frege 1885*; an earlier translation of Frege's response by Hans Kaal is published in *McGuinness 1984*, 122.

[3] Compare *Thiel 1986*, LII.

[4] Compare *Frege 1884*, 80 fn, and Frege's remark in the last sentence of his reply to Cantor.

[5] See *Rang and Thomas 1981*.

[6] Russell's characterisation of the paradox in his *1902* letter is, in fact, not well-formed in Frege's system. However, Frege himself, in his *1902* reply to Russell, 213, provides the proper reformulation of the paradox.

Tait goes even further and deems Frege 'reckless' for having missed Cantor's *explicit* warning regarding the notion of extension. He writes:

> 'But in fact his assumption in the *Grundgesetze* that every concept has an extension was an act of recklessness, forewarned against by Cantor already in 1883 and again, explicitly, in his review in 1885.' *Tait 1997*, 248.

As such, Cantor's purported inkling would have predated the discovery of the Russell-Zermelo paradox by almost two decades.

In what follows, we will discuss this alleged implicit (or even explicit) warning. We will separate two issues: first, whether the most natural reading of Cantor's criticism provides an indication that the notion of extension is defective; second, whether there are other ways of understanding Cantor that support such an interpretation and can serve as a precisification of Cantor's presumed warning.

1 The natural reading of Cantor's criticism

The most natural reading of Cantor's criticism coincides with Frege's and Zermelo's understanding of the relevant passages. Cantor accuses Frege of being subject to a vicious circularity. That is, in order to be in a position to employ extensions in the manner envisaged, the concept *number* has to have been given independently and in advance. The crucial section reads as follows:

> [Frege] overlooks altogether the fact that the 'extension of a concept' is, in general, something quantitatively completely indeterminate; only in certain cases is the 'extension of a concept' quantitatively determinate, in such a case however, if it is finite, a determinate number [*Zahl*] belongs to it, and if it is infinite, a determinate cardinality [*Mächtigkeit*]. For such a quantitative determination of the 'extension of a concept', however, the concepts 'number' and 'cardinality' have to be already given independently in advance, and it is an *inversion of what is correct* to attempt to ground the latter concepts on the concept 'extension of a concept'.

Let us characterise the argument in the following way: (1) The extension of a concept is not always quantitatively determinate. (2) For the extension of a concept to be quantitative determinate, a determinate number or determinate cardinality has to belong to it. Therefore, (3) the concept *number* (or *cardinality*) has to be given independently and in advance. Hence, (4) the attempt to reduce the concept *number* to the concept *extension of a concept* reverses the proper order of reduction.

Cantor provides no further discussion of the premises of his argument and, indeed, Frege does not disagree with Cantor at this point. Rather, Frege as well as Zermelo sees the principal mistake of Cantor's concern in a crucial misunderstanding of how Frege defines 'cardinal number'[7] with the aid of extensions. Frege contends (and Zermelo agrees) that the validity of the argument depends on this misunderstanding and that the definition he actually employs is not subject to Cantor's criticism.

Frege defines the cardinal number of F as the extension of the concept *equinumerous with the concept F*. Cantor's objection could, and surely would, be a genuine concern, as Frege admits in his response, if the cardinal number of F were instead defined as the extension of the concept F. In that case, a quantitative indeterminacy of the extension of F would render the definition of cardinal number of F improper, since no determinate number would be defined. In contrast, on Frege's actual definition a quantitative indeterminacy of the concept *equinumerous with the concept F* does not create similar obstacles. How many concepts are equinumerous with F is irrelevant to the definition of the cardinal number of F.

To return to our main question regarding Cantor's alleged warning, there is no hint concerning the defectiveness of extensions in general on this most natural reading of Cantor's argument. In fact, his discussion seems to presuppose that there is nothing in principle wrong with the notion of extension. After all, Cantor is concerned with the order of priority of the concepts *extension* and *cardinal number*, and such considerations concerning the order of priority hardly involve defective concepts.

2 Alternative readings of Cantor's criticism

Readings that might lead to the interpretation of Cantor as providing a hint or warning concerning the defectiveness of extensions emphasise the first premise of the argument, i.e. that 'the "extension of a concept" is, in general, something quantitatively completely indeterminate'. In other words, there are some extensions that are not quantitatively determinate. This is then understood as a warning to Frege concerning such quantitatively indeterminate extensions.

What could this warning be in detail? Let us first exclude some potentially tempting candidates. Cantor cannot be interpreted as saying that quantitative determinacy of an extension is a necessary conditions for there being an extension. After all, Cantor says explicitly that there are extensions that are not quantitatively determinate. Furthermore, he surely did not intend to convey that some concepts do not have an extension. The distinction that Cantor emphasises is between extensions that are quantitatively determinate and those that are not; this is orthogonal to the question whether each concept has an extension and would thus be a rather obscure

[7] In the translations we use 'number' as a translation of both 'Zahl' and 'Anzahl' providing the German parenthetically in the text. 'Anzahl' should be translated as 'cardinal number' in Frege; Cantor, however, understands 'Anzahl' as ordinal number. Later parts of his criticisms are based on just this confusion. In order not to render this confusion even more perplexing, we opted for the neutral term 'number' in the translation. In our discussion here, we use 'cardinal number' instead for the sake of precision.

way of making the point.[8] Finally, the warning might be thought to be that the concept *extension* in general is inconsistent. This first raises the question why Cantor did not say so explicitly but rather vaguely insinuates the matter in such a roundabout way. Moreover, the whole ensuing discussion would be hard to explain. As mentioned above, Cantor argues that conceptually cardinal number is prior to extensions which would make no sense if the concept *extension* were inconsistent.

A more promising candidate for the alleged warning is interpreting Cantor as covertly suggesting that extensions that are not quantitatively determinate might lead into inconsistency. A reading along these lines would make sense of the rest of the argument and the priority claim mentioned above. Namely, if Cantor is to be understood in such a way that quantitatively indeterminate extensions might lead into inconsistency, then this would further emphasise the importance of the concept *number* and its priority. The concept *number* has to be prior to that of *extension* since it is only by its means that we can recognise which extensions are the 'good' ones. 'Bad' extensions, on the other hand, lead to the now well-known paradoxes.

First to note is that on this reading, Frege's (and Zermelo's) response would be missing the point. For Cantor's argument, so understood, does not rely on any specific definition of 'cardinal number' but rather concerns the use of extensions in such a definition in any form. Secondly, the interpretation of the passage as containing a warning of a lurking inconsistency — despite the fact that Cantor does not mention inconsistency — draws on the knowledge we now have of the contradiction and the connection that is commonly seen today between the set-theoretic paradoxes and collections that are somehow 'too big'. Specifically, 'quantitatively indeterminate' would have to be understood as something akin to 'too big to form a set'. This, however, is an *ex post* explanation of Cantor's intention and a mistake. There is, first of all, no textual evidence in the review that Cantor understood quantitative indeterminacy as a notion that pertains to the size of an extension. In fact, note that in his review Cantor always mentions finite extensions alongside infinite extensions in his discussion of quantitative indeterminacy. This supports the conclusion that Cantor did not regard quantitative indeterminacy as a feature of only infinite extensions. An interpretation of quantitative indeterminacy as 'too big' is thus in direct tension with this observation and merely feeds off our more recent knowledge of the paradoxes and their connection to size. Without the understanding of 'quantitatively indeterminate' as 'too big', we are then, it seems, with no clear alternative of how to interpret Cantor as insinuating that quantitative indeterminacy leads to inconsistency.

One question remains: is there a reading of Cantor's argument, whether or not intended by him, that (a) does not rely on a specific misunderstanding of Frege's definition of cardinal number, i.e. where there is no specific assumption about the role extensions play in the definition, and (b) renders the argument valid. One such

[8] This is in sharp contrast to Tait's claim that Cantor explicitly warns against Frege's assumption that every concept has an extension. There is simply no *explicit* warning of this kind. Below we discuss a reading of Cantor that could be regarded as an *implicit* warning to this effect.

interpretation might be the one we just encountered where quantitative determinacy is connected to size. However, it fails to render the argument valid. The resources of second-order logic suffice, as is now well-known, to rule out extensions that are too big to form a set;[9] no appeal to numbers or cardinalities is necessary.

Here is not the space further to indulge in speculations about possible rescue attempts of Cantor's argument to the claim that the concept *number* is prior to the concept *extension*, or even alternative arguments for his intriguing conclusion. Let us just note that the sought-after reading of Cantor's argument has to present a novel route into inconsistency that is not connected to size. To reiterate, it seems implausible that Cantor should have been aware of such a problem without ever mentioning it explicitly.

Translations

Cantor's review of Frege

Published in German in *Deutsche Litteraturzeitung*, **VI**, no. 20, 16 May 1885, columns 728–29. Reprinted in *Cantor 1932*, 440–41; and in *Thiel 1986*, 117–19.[10]

The purpose of this pamphlet, to subject the foundations of arithmetic to a renewed investigation, is a laudable one. For there is no doubt that this branch of mathematics, which serves as a basis for all other mathematical disciplines, demands a far deeper exploration of its basic concepts and methods than it has generally received so far. It has also to be acknowledged that the author has adopted the right point of view in putting forward the requirement that both spatial and temporal intuition, as well as all psychological aspects, have to be kept at a distance from arithmetical concepts and principles. For it is only in this way that their rigorous logical purity, and thereby also a entitlement to apply the arithmetical tools to objects of intuitive knowledge, can be achieved.

Assuming this standpoint, the author dedicates by far the most space to a critical examination of previous attempts to provide a foundation of arithmetic; the objections he puts forward against the doctrines of Kant, Stuart Mill and others, are mostly correct and can be recommended for attention.

Less successful, in contrast, seems to me to be his own attempt at a rigorous foundation of the concept of number. The author arrives — seemingly following a suggestion by Ueberweg in his 'System der Logik',[11] §53 — at the unfortunate idea of taking what is called the 'extension of a concept' in school logic as the foundation of the concept of number; he overlooks altogether the fact that the 'extension of a concept' is, in general, something quantitatively completely indeterminate; only in certain cases is the 'extension of a concept' quantitatively determinate, in such a case

[9] See *Boolos 1987*, *Boolos 1989* and also *Cook 2003*.

[10] Zermelo's edition, *Cantor 1932*, adds various emphases. We here follow the original publication.

[11] See *Ueberweg 1857/*^{5}*1882*.

however, if it is finite, a determinate number [*Zahl*[12]] belongs to it, and if it is infinite, a determinate cardinality [*Mächtigkeit*]. For such a quantitative determination of the 'extension of a concept', however, the concepts 'number' and 'cardinality' have to be already given independently in advance, and it is an *inversion of what is correct* to attempt to ground the latter concepts on the concept 'extension of a concept'.

If this point has escaped the author, then this probably has to be attributed to the circumstance that his principal mistake is well hidden indeed, concealed under the veil of his most subtle distinctions.

Accordingly, I also do not regard it as correct when the author expresses the opinion in §85 that what I call 'cardinality [*Mächtigkeit*]' coincides with what he calls 'number [*Anzahl*]'. I call 'cardinality of a | collection [*Inbegriff*] or of a set [*Menge*] of elements' (where the latter can be homogenous or heterogeneous, simple or composite) that general concept under which fall all and only those sets that are equivalent to the given set. Here, two sets are to be called 'equivalent', if they can be correlated one-to-one with each other, element for element.

What I call 'number [*Anzahl*] or ordinal number [*Ordnungszahl*]' is something different; I assign it only to 'well-ordered sets', that is, I understand by 'number [*Anzahl*] or ordinal number of a well-ordered set' that general concept under which fall all and only well-ordered sets that are similar to the one given. I call two well-ordered sets 'similar' if, element for element, they can be mapped one-to-one onto each other, in such a way as to respect the given sequence of the elements on both sides. For finite sets, the two aspects 'cardinality' and 'number' coincide, as it were, since a finite set has one and the same ordinal number in every arrangement of its elements as a 'well-ordered set'; for infinite sets, in contrast, the difference between 'cardinality' and 'ordinal number' comes to light most strikingly, as was clearly shown in my pamphlet, '*Grundlagen einer allgemeinen Mannigfaltigkeitslehre*', Leipzig 1883.[13]

What the author says against my use of the word 'number [*Anzahl*]' hardly seems justified; he appeals to the popular use of language which must have no authority at all when fixing scientific concepts, in the present context, however, where it surely only refers to finite sets, it should hardly be harmed by my precisification of the concept of number [*Anzahlbegriffs*].

Halle a. S.

Georg Cantor

Editorial Note by Zermelo

In his edition of Cantor's *Gesammelte Abhandlungen*, 1932, 141–42. Reprinted in *Frege, 1986*, 119.

[12]See our footnote 7 above.

[13]See *Cantor 1883*.

Frege's book is today more and more recognised, and at least in the opinion of the present editor provides perhaps the best and clearest account of the concept of number [*Anzahl*] published on the subject at all so far. It is however only partly done justice to by Cantor in his review. In fact, Frege understands by 'number [*Anzahl*]' exactly what Cantor denotes by 'cardinal number' [*Kardinalzahl*], namely the invariant, that what is common to all equivalent (Frege says 'equinumerous') sets (Frege says 'concepts'). It is just that Frege identifies the class-invariant with the 'extension of the concept: equinumerous with the concept F'. This extension of a concept, however, is nothing but a logical 'class', specifically, it is the class of the 'sets' equivalent to F or 'concepts' [equinumerous to] F. So by no means does it have to be 'quantitatively determinable', for it is not to it, but rather to the concept F itself, that the predicate 'number [*Anzahl*]' is applied. The introduction of the 'extension of a concept' may surely, as Frege himself admits, be attended by its own disadvantages and misgivings, but it is basically inessential, and Cantor's criticism here seems to rest on a misunderstanding. On the other hand, Cantor was surely justified in introducing *his* concept of 'number [*Anzahl*]' as of an order type for *transfinite* sets, which Frege does not even consider. For us today it can only seem striking and regrettable that the two contemporaries, the great mathematician and the commendable logician, have, as this review shows, understood each other so little.

Frege's Reply to Cantor

Gottlob Frege: 'Erwiderung auf Georg Cantors Rezension der *Grundlagen*', in: *Deutsche Litteraturzeitung*, **VI**, no. 28, 11 July 1885, column 1030. Reprinted in *Frege 1986*, 120.

In his review of my *Grundlagen der Arithmetik* in no. 20 of this journal, Mr Cantor notes that it is only in certain cases that the extension of a concept is quantitatively determinate; then, however, in the finite case, a determinate number [*Zahl*] belongs to it; for such a determination, however, the concept 'number [*Zahl*]' must already be given independently. These remarks would be very apt and I would regard them as altogether legitimate if it were a consequence of my definition that, e.g., the number [*Anzahl*] of the moons of Jupiter is the extension of the concept 'moon of Jupiter'. They are not at all apt, however, for the definition I have given, according to which the number [*Anzahl*] of the moons of Jupiter is the extension of the concept 'equinumerous with the concept "moon of Jupiter" '; for the quantitative determination of the extension of this concept is of no concern here. — I thus surmise that there is a misunderstanding, which I hereby wish to have eliminated. The unfavourable judgement in the third paragraph of the review thus also lapses. With the same right, or rather lack thereof, one might just as well object to the explanation given by Mr Cantor in his fifth paragraph: a general concept is, in general, something quantitatively completely indeterminate. In certain circumstances, however, a number [*Zahl*] belongs to it. For such a quantitative determination, however, the concept 'number [*Zahl*]' has to be already given independently. The difference, that Mr Cantor says 'general concept' where I say 'extension of a concept' appears, incidentally, insignificant considering the note of p. 80 of my book.

Jena

G. Frege

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References

Boolos, G. 1987. 'Saving Frege from Contradiction', *Proceedings of the Aristotelian Society*, **87**, 137–51.

Boolos, G. 1989. 'Iteration Again', *Philosophical Topics*, **42**, 5–21.

Cantor, G. 1883. *Grundlagen einer allgemeinen Mannigfaltigkeitslehre. Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen*, Leipzig: Teubner; reprinted in *Cantor 1932*, 165–208; translated by William Ewald in his (ed.) *From Kant to Hilbert: a source book into the foundations of mathematics*, Oxford: Clarendon Press, 1996, 878–920.

Cantor, G. 1885. Review of *Frege 1884*, *Deutsche Litteraturzeitung*, **VI**, no. 20, 16 May, columns 728–29; reprinted in *Cantor 1932*, 440–41, and in *Thiel 1986*, 117–19.

Cantor, G. 1932. *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, edited and with annotations by Ernst Zermelo. Berlin: Springer.

Cook, R. T. 2003. 'Iteration One More Time', *Notre Dame Journal of Formal Logic*, **44**, 63–92.

Frege, G. 1884. *Die Grundlagen der Arithmetik. Eine logisch-mathematische Untersuchung über den Begriff der Zahl*, Breslau: W. Koebner, reprinted in *Thiel 1986.*

Frege, G. 1885. 'Erwiderung auf Georg Cantors Rezension der *Grundlagen*', *Deutsche Litteraturzeitung*, **VI**, no. 28, 11 July, column 1030; reprinted in *Frege 1967*, 112, and *Thiel 1986*, 120; translated by Hans Kaal in *McGuinness 1984*,122.

Frege, G. 1893/1903. *Grundgesetze der Arithmetik*, 2 vols, Jena: H. Pohle.

Frege, G. 1902. Letter to Russell, 22 June, published in *Frege 1976*, 212–15.

Frege, G. 1967. *Kleine Schriften*, edited by Ignacio Angelelli, Hildesheim: Olms.

Frege, G. 1976. *Wissenschaftlicher Briefwechsel*, edited by Gottfried Gabriel, Hans Hermes, Friedrich Kambartel, Christian Thiel and Albert Veraart. Hamburg: Meiner.

McGuinness, B., editor, 1884. *Gottlob Frege: Collected Papers on Mathematics, Logic, and Philosophy*, Oxford: Basil Blackwell.

Rang, B. and Thomas, W. 1981. 'Zermelo's Discovery of the "Russell Paradox" ', *Historia Mathematica*, **8**, 15–22.

Russell, B. 1902. Letter to Frege, 16 June, published in *Frege 1976*, 211–12.

Tait, W. W. 1997. 'Frege versus Cantor and Dedekind: On the Concept of Number', in W. W. Tait, *Early Analytic Philosophy*, Chicago: Open Court, 213–48.

Thiel, C., editor, 1986. *Gottlob Frege: Die Grundlagen der Arithmetik. Eine logisch-mathematische Untersuchung über den Begriff der Zahl. Centenarausgabe. Mit ergänzenden Texten*, critical centennial edition of *Frege 1884* with additional texts, Hamburg: Meiner.

Ueberweg, F. 1857/ ^{5}1882. *System der Logik und Geschichte der logischen Lehren*. First edition 1857, fifth, revised edition provided by Jürgen Bona Meyer, Bonn: Adolph Marcus, 1882; authorised translation into English based on the third edition (1868) with notes and appendices by Thomas M. Lindsay as: *System of Logic and History of Logical Doctrines*. London: Longmans, Green and Co., 1871.

Zermelo, E. 1932. Editorial note to *Cantor 1885*, in *Cantor 1932*, 141–42; reprinted in *Thiel 1986*, 119.